Improving a Family of Approximation Algorithms to Edge Color Multigraphs

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Abstract

Given a multigraph $G = (V, E)$, the *Edge Coloring Problem* (ECP) calls for the minimum number χ of colors needed to color the edges in E so that all edges incident with a common node are assigned different colors. The best known polynomial time approximation algorithms for ECP belong to a same family, which is likely to contain, for each positive integer k, an algorithm which uses at most $\lceil((2k+1)\chi + (2k-2))/2k\rceil$ colors. For $k \leq 5$ the existence of the corresponding algorithm was shown, whereas for larger values of k the question is open. We show that, for any k such that the corresponding algorithm exists, it is possible to improve the algorithm so as to use at most $\lceil((2k+1)\chi + (2k-3))/2k\rceil$ colors. It is easily shown that the $(2k-3)/2k$ term cannot be reduced further, unless $P = NP$. We also discuss how our result can be used to extend the set of cases in which well-known conjectures on ECP are valid.

Key words: Edge Coloring, Approximation Algorithm, Matching.

1 Introduction

Given a multigraph $G = (V, E)$, the *Edge Coloring Problem* (ECP) calls for coloring the edges in E by using as few colors as possible so that all edges incident with a common node are assigned different colors. Let χ denote the optimal solution value of ECP, i.e. the minimum number of colors required, and Δ be the maximum degree of a node in V. Clearly, $\chi \geq \Delta$. Moreover, if G is simple, i.e. it does not contain parallel edges, a basic result of Vizing [14] states that $\chi \leq \Delta + 1$. On the other hand, Holyer showed in [7] that ECP is NP-hard even for simple graphs for which $\Delta = 3$, and hence $\chi = 3$ or 4. Therefore, unless $P = NP$, the algorithmic proofs of Vizing's result (the fastest algorithm is due to Gabow, Nishizeki, Kariv, Leven and Terada [3]) constitute best possible, in terms of worst-case behavior, polynomial-time approximation algorithms for ECP on simple graphs, namely algorithms which are guaranteed to return a solution of value within one unit of the optimum.

Unfortunately, many relevant applications of ECP are associated with nonsimple graphs (see e.g. Fiorini and Wilson [2]). For general graphs, it is easy to see that the difference between χ and Δ can be arbitrarily large: Consider for instance a multitriangle with three nodes and k parallel edges between each pair of nodes, for which $\chi = 3k$ and $\Delta = 2k$. Nevertheless, it is common belief that there exists a polynomial-time algorithm for ECP on multigraphs which

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is guaranteed to return a solution within one unit of the optimum, even if such an algorithm is unknown so far. At present, the best known polynomial-time approximation algorithm for ECP is due to Nishizeki and Kashiwagi [9], and is guaranteed to color the edges of a multigraph by using at most $\lfloor (11\chi + 8)/10 \rfloor$ colors. This algorithm improves on previous algorithms of Andersen [1], Nishizeki and Sato [10], Goldberg [5] and Hochbaum, Nishizeki and Shmoys [6], which are all based on similar approaches. Whereas it would be conceptually clear how one should try to generalize the main idea of these algorithms, so as to achieve a solution using at most $\chi + 1$ colors (see [6]), the proof techniques used in [1, 10, 5, 6, 9] require such a complicate case analysis that probably any future improvement on the 11/10 factor will require some alternative approach. This is probably the reason why the algorithm in [9], which is more than 10 years old, is still the best one known for a widely-studied problem as ECP.

In this paper, we show how to reduce to 7/10 the constant term 8/10 in the performance of the algorithm in [9], obtaining an algorithm for ECP using at most $\lfloor (11\chi + 7)/10 \rfloor$ colors. It is easily shown that the 7/10 term cannot be reduced further, unless $P = NP$. More generally, we show a technique to achieve an ECP solution of value at most $\lfloor \alpha\chi + \gamma \rfloor$ when an approximation algorithm using at most $\max\{\lfloor \alpha\Delta + \beta \rfloor, \chi\}$ colors is available, where $\gamma = \max\{\beta + 1 - \alpha, 4 - 3\alpha\}$. Even if our improvement is not impressive, it is the first one on the algorithm of [9], almost 8 years after its publication. Moreover, our method is applicable also to possible future algorithms belonging to the same family as those in [1, 10, 5, 6, 9]. We also discuss how our result can be used to extend the set of cases in which well-known conjectures on ECP formulated by Seymour [12] and Goldberg [5] are valid.

We conclude this section with the basic definitions and notation used in the rest of the paper. Let $G = (V, E)$ be a multigraph for which the maximum degree of a node is Δ and the optimal ECP solution has value χ. For convenience, and without loss of generality, we suppose that G is connected. A *matching* of G is an edge set $M \subset E$ such that each node of G is the endpoint of at most one edge in M. If each node in $S \subseteq V$ is the endpoint of some edge in M then we say that M is an *S-matching*. Given a matching M, the graph $G \setminus M$ is the one with node set V and edge set $E \setminus M$. Clearly, any ECP solution using ν colors defines a partition of the edge set E into ν matchings, here denoted by $C_1, \dots, C_\nu$, each corresponding to the edges which receive a same color. Given a node set $S \subseteq V$, we let $\delta(S)$ denote the set of edges with exactly one endpoint in S and $E(S)$ denote the set of edges with both endpoints in S. Moreover, the graph $G \setminus S$ is the one with node set $V \setminus S$ and edge set $E(V \setminus S)$. By *connected component* of a graph we mean a node set $T \subseteq V$ such that $\delta(T) = \emptyset$ and $\delta(S) \neq \emptyset$ for any $S \subset T$.

2 The improvement

Let us consider a polynomial-time approximation algorithm for ECP, called *Approx*, guaranteed to return a solution which uses at most $\max\{\lfloor \alpha\Delta + \beta \rfloor, \chi\}$ colors. We assume that $\Delta \geq 3$, otherwise ECP is easily solved, and $\alpha \leq 4/3$, since there exist algorithms capable of edge-coloring a multigraph using at most $\max\{\lfloor 4\Delta/3 \rfloor, \chi\}$ colors. For example, the algorithm of [9] uses at most $\max\{\lfloor (11\Delta + 8)/10 \rfloor, \chi\}$ colors.

Let $X \subseteq V$ be the set of nodes of G having degree Δ. The basic property used by our improvement is the following

Lemma 1 *If G does not contain any X-matching, then $\chi \geq \Delta + 1$.*

Proof. Assume $\chi = \Delta$ and let $C_1, \ldots, C_\Delta$ be a Δ-edge coloring of G, i.e. a partition of the edges of G into Δ matchings. Then C_i is an X-matching for $i = 1, \ldots, \Delta$ and we have a contradiction. □

Our algorithm, called *Impr*, first checks for the existence of an X-matching M. If such an M exists, algorithm *Approx* is used to color the edges of $G \setminus M$, and then an additional color is used for the edges in M. Otherwise, algorithm *Approx* is applied to color the edges of G. The algorithm can be sketched as follows.

Impr(G) *require:* G is a multigraph with $\Delta \geq 3$

If G contains no X-matching, then return the coloring returned by *Approx*(G).

Let M be an X-matching of G. Let $(C_1, \ldots, C_k)$ be the coloring returned by *Approx*$(G \setminus M)$. Return the coloring $(C_1, \ldots, C_k, M)$.

Proposition 1 *By using an algorithm Approx which is guaranteed to find an ECP solution using at most* $\max\{\lfloor \alpha\Delta + \beta \rfloor, \chi\}$ *colors, algorithm Impr is guaranteed to find an ECP solution using at most* $\lfloor \alpha\chi + \gamma \rfloor$ *colors, where* $\gamma = \max\{\beta + 1 - \alpha, 4 - 3\alpha\}$.

Proof. Suppose first an X-matching M exists, and let $G' = G \setminus M$. In G', the maximum degree Δ' of a node equals $\Delta - 1$. Let χ' denote the optimal solution value of ECP on G'. *Approx* returns a solution using at most $\max\{\lfloor \alpha\Delta' + \beta \rfloor, \chi'\}$ colors. If the number of colors used is at most $\lfloor \alpha\Delta' + \beta \rfloor$, then the number of colors used by *Impr* to color G is at most $\lfloor \alpha\Delta' + \beta \rfloor + 1 = \lfloor \alpha\Delta' + \beta + 1 \rfloor = \lfloor \alpha\Delta + \beta + 1 - \alpha \rfloor$. Otherwise, the number of colors used by *Approx* is χ', and then the number of colors used by *Approx* is $\chi' + 1 \leq \chi + 1$. The relation $1 + \chi \leq \lfloor \alpha\chi + \gamma \rfloor$ is equivalent to $1 + \chi \leq \alpha\chi + \gamma$, i.e. $\gamma \geq 1 + (1 - \alpha)\chi$. Since $\chi \geq 3$ and $\alpha \geq 4/3$, the latter inequality is satisfied if and only if $\gamma \geq 4 - 3\alpha$ holds.

Now suppose that no X-matching exists. By Lemma 1, $\chi \geq \Delta + 1$. Therefore, if *Approx* returns a solution using at most $\lfloor \alpha\Delta + \beta \rfloor$ colors, then the number of colors used by *Impr* is at most $\lfloor \alpha(\chi - 1) + \beta \rfloor = \lfloor \alpha\chi + \beta - \alpha \rfloor$. Otherwise, *Approx* returns a solution using at most χ colors, i.e. an optimal one. □

As a corollary, by using as algorithm *Approx* the one presented in [9], for which $\alpha = 11/10$ and $\beta = 8/10$, *Impr* edge colors a graph by using at most $\lfloor (11\chi + 7)/10 \rfloor$ colors. As already mentioned, assuming $P \neq NP$, by [7] any polynomial-time approximation algorithm for ECP may return a solution of value 4 when $\chi = 3$. Hence, if the algorithm delivers a solution using at most $\lfloor \alpha\chi + \gamma \rfloor$ colors, the relation $\gamma \geq 4 - 3\alpha$ must hold. This shows that $7/10$ is the best possible value for γ as long as $\alpha = 11/10$.

By using standard matching reduction techniques (see e.g. Gerards [4]) the determination of an X-matching, if any, can be carried out by finding a matching of maximum cardinality in the graph $\tilde{G} = (\tilde{V}, \tilde{E})$ obtained by taking two identical copies of G, say $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and, for each node $v \in V \setminus X$, by introducing a new edge $(v_1, v_2) \in \tilde{E}$, where v_1 and v_2 are the counterparts of v in V_1 and V_2, respectively. Clearly G has an X-matching if and only if $\tilde{G}$ has a perfect matching. Moreover, the complexity of finding a perfect matching of $\tilde{G}$, if any, is $O(|\tilde{V}|^{1/2}|\tilde{E}|)$ (see [8]), i.e. $O(|V|^{1/2}|E|)$, since $|\tilde{V}| = 2|V| = O(|V|)$ and $|\tilde{E}| = 2|E| + |V \setminus X| = O(|E|)$ (remember that G is connected). Accordingly, our improvement does not increase the asymptotic $O(|E|(|V| + \Delta))$ running time of the algorithm in [9] and the others in the family [1, 10, 5, 6].

The family of algorithms including that of [9] and previous ones, [1, 10, 5, 6], is likely to contain, for each positive integer k, an algorithm for which the number of colors needed is at most

$$\max\left\{\left\lceil\frac{(2k+1)\Delta + (2k-2)}{2k}\right\rceil, \chi\right\}.$$

For $k \leq 5$, the existence of the corresponding algorithms was shown, while for larger values of k the question is open. Anyway, if the algorithm corresponding to some other value of k turns out to exist, our technique will immediately provide an approximation algorithm requiring at most

$$\left\lceil\frac{(2k+1)\chi + (2k-3)}{2k}\right\rceil$$

colors, decreasing the constant term in the worst-case performance bound to its minimum possible value (unless $P = NP$).

3 Stronger Results

For a multigraph $G = (V, E)$, define Γ as the maximum value of $\lceil 2|E(S)|/(|S| - 1)\rceil$ computed over all subsets $S \subseteq V$ with $|S|$ odd. It is easy to check that $\chi \geq \Gamma$, since each color can be assigned to at most $(|S| - 1)/2$ edges in $E(S)$ if $|S|$ is odd. Accordingly, $\Phi = \max\{\Delta, \Gamma\}$ is a lower bound on the optimal ECP solution value. A well-known conjecture formulated by Seymour [12] and Goldberg [5] states that for every graph $\chi \leq \max\{\Delta + 1, \Gamma\}$. A weaker form of this conjecture (see Seymour [12]) reads

Conjecture 1 *For every graph,* $\chi \leq \Phi + 1$.

Conjecture 1 sounds very interesting also from an algorithmic point of view since the value of Φ can be determined in polynomial time (see Padberg and Wolsey [11]).

Already in [12] Seymour claimed that the conjecture was true for $\Phi \leq 6$. The algorithm presented in [9] returns a solution which uses in fact no more than $\max\{\lfloor(11\Delta + 8)/10\rfloor, \Gamma\}$ colors, showing that Conjecture 1 holds for $\Phi \leq 11$. In this section, we show that our method guarantees finding a solution using at most $\lfloor(11\Phi + 7)/10\rfloor$ colors, extending to $\Phi \leq 12$ the known range of validity of the conjecture. Again, our method is applicable to possible future improvements on the algorithm in the family of [9], as discussed in the previous section. In particular, before our result, improving from $(2k+1)/2k$ to $(2k+3)/(2k+2)$ the coefficient for χ in the worst-case performance bound of the best ECP algorithm allowed one to extend the range of validity of the conjecture from $\Phi \leq 2k+1$ to $\Phi \leq 2k+3$. Now, the same improvement results into an extension of the range of validity from $\Phi \leq 2k + 2$ to $\Phi \leq 2k + 4$.

Suppose the polynomial-time approximation algorithm *Approx* mentioned in the previous section is guaranteed to find a solution which uses at most $\max\{\lfloor \alpha\Delta + \beta \rfloor, \Gamma\}$ colors. Again, assume $\Delta \geq 3$ and $\alpha \leq 4/3$ and let $X \subseteq V$ be the set of nodes in V having degree Δ. We start by proving a stronger version of Property 1.

An X-*Tutte prover* for G is a node set $S \subset V$ such that $G \setminus S$ has strictly more than $|S|$ connected components of odd cardinality containing only nodes in X. The following result, which is essentially a reformulation of the well-known Tutte characterization of graphs not having a perfect matching (see e.g. [4]), was pointed to our attention by Seymour [13].

Lemma 2 *G contains an X-matching if and only if it does not contain an X-Tutte prover.*

Proof. Clearly, if G contains an X-Tutte prover it does not contain an X-matching. To prove the converse implication, consider the graph $\overline{G} = (\overline{V}, \overline{E})$ obtained from $G = (V, E)$ as follows. If $|V|$ is odd, let $\overline{V} = V \cup \{t\}$, where t is an additional dummy node, otherwise let $\overline{V} = V$. Moreover, let $\overline{E} = E \cup \{(u,v) : u, v \in \overline{V} \setminus X\}$. Since all nodes in $\overline{V} \setminus X$ are pairwise connected and $|\overline{V}|$ is even, G contains an X-matching if and only if $\overline{G}$ contains a perfect matching. By Tutte's characterization, $\overline{G}$ has a perfect matching if and only if it does not contain a node set $\overline{S} \subset \overline{V}$ such that $\overline{G} \setminus \overline{S}$ has $p > |\overline{S}|$ connected components $\overline{T}_1, \ldots, \overline{T}_p$ such that $|\overline{T}_i|$ is odd for $i = 1, \ldots, p$. Suppose $\overline{G}$ contains such an $\overline{S}$ and then no perfect matching, i.e. G contains no X-matching. Since $\overline{G} \setminus X$ is a complete graph, at most one component among $\overline{T}_1, \ldots, \overline{T}_p$ can contain some node not in X. Furthermore, $|\overline{T}_1| + \ldots + |\overline{T}_p| + |S|$ is even, meaning that p and $|S|$ have the same parity and hence $p \geq |S| + 2$. Let S be equal to $\overline{S} \setminus \{t\}$ if $|\overline{V}|$ is even and to $\overline{S}$ otherwise: It is immediate to check that $G \setminus S$ has at least $p - 1 > |S|$ connected components of odd cardinality containing only nodes in X, i.e. S is an X-Tutte prover. □

The above variant of Tutte's characterization allows us to prove

Lemma 3 *If G does not contain any X-matching, then $\Gamma \geq \Delta + 1$.*

Proof. By Lemma 2, if G does not contain an X-matching, then it contains an X-Tutte prover S. Let $T_1, \ldots, T_p$, $p > |S|$, be the connected components of $G \setminus S$ with $|T_i|$ odd and $T_i \subseteq X$ for $i = 1, \ldots, p$. We show that $2|E(T_i)|/(|T_i| - 1) > \Delta$ for some i. Suppose indeed this is false, i.e. $2|E(T_i)|/(|T_i| - 1) \leq \Delta$ for $i = 1, \ldots, p$. As all nodes in $T_1, \ldots, T_p$ have degree Δ, $\Delta|T_i| = 2|E(T_i)| + |\delta(T_i)|$, hence $|\delta(T_i)| \geq \Delta$ for $i = 1, \ldots, p$. Therefore, $|\bigcup_{i=1}^{p} \delta(T_i)| \geq p\Delta > |S|\Delta$, which is a contradiction since all edges in $\bigcup_{i=1}^{p} \delta(T_i)$ have an endpoint in S and all nodes in S have degree $\leq \Delta$. □

We can then prove a statement analogous to Proposition 1, by following exactly the same proof, using Lemma 3 instead of Lemma 1.

Proposition 2 *By using an algorithm Approx which is guaranteed to find an ECP solution using at most $\max\{\lfloor \alpha\Delta + \beta \rfloor, \Gamma\}$ colors, algorithm Impr is guaranteed to find an ECP solution using at most $\lfloor \alpha\Phi + \gamma \rfloor$ colors, where $\gamma = \max\{\beta + 1 - \alpha, 4 - 3\alpha\}$.*

To check that Conjecture 1 holds for $\Phi \leq 12$, just observe that the solution returned by algorithm *Impr* applied by using algorithm *Approx* of [9] may use strictly more than $\Phi + 1$ colors only if $\lfloor (11\Phi + 7)/10 \rfloor \geq \Phi + 2$, i.e. $\Phi \geq 13$.

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