

# König's Edge Coloring Theorem without augmenting paths

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We give a simple, self-contained proof of the following basic fact [1, 2] in matching theory:

**Theorem** *Every bipartite regular multigraph is factorizable.*

*Proof:* Assume  $G$  to be a counterexample with the smallest number of edges. Then  $G$  is  $r$ -regular for some integer  $r \geq 1$ . Let  $e = uv$  be any edge of  $G$ . In  $G$ , remove nodes  $u$  and  $v$ , then add a set of edges  $F$  as to obtain a bipartite multigraph  $G'$  which is  $r$ -regular. Note that  $|F| < r$  and  $G'$  has less edges than  $G$ . By assumption  $G'$  contains  $r$  disjoint 1-factors. Since  $|F| < r$ , at least one of these 1-factors, say  $M'$ , is disjoint from  $F$ . Therefore  $M = M' \cup \{e\}$  is a 1-factor of  $G$ . Moreover  $G \setminus M$  has less edges than  $G$  and is factorizable. But then  $G$  is factorizable.  $\square$

Note that the above proof does not use any alternating path argument.

## References

- [1] D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916) 453–465.
- [2] D. König, Graphok és alkalmazásuk a determinánsok és a halmazok elméletére, Math. Termész. Ért. 34 (1916) 104–119.