Minimum T-cuts and optimal T-pairings

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Abstract

We introduce the notion of T-pairing and give a min-max characterization for the minimum size of a T-cut. We show that the coefficients in the minimal TDI system for the T-cut polyhedron can be arbitrarily big.

Key words: T-cut, minimum T-cut, T-pairing, Gomory-Hu tree.

1 Introduction

Given a connected graph $G = (V, E)$ and $S \subseteq V$, the *cut* $\delta_G(S)$ (subscripts omitted when no confusion arises) is the set of those edges in E with precisely one endnode in S. When $T \subseteq V$ has even cardinality, the pair (G, T) is called a *graft*. A T-*cut* is a cut $\delta(S)$ such that $|S \cap T|$ is odd. In the following, $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of non-negative reals and the set of non-negative integers, respectively. Given a *cost* function $c : E \mapsto \mathbb{R}_+$, the cost $c(F)$ of a set of edges F is defined as $\sum_{e \in F} c_e$. Denote by $\lambda_{G,T,c}$ the minimum cost of a T-cut in (G, T, c). When, as a special case, $c = \mathbf{1}$, then we denote by $\lambda_{G,T}$ the minimum size of a T-cut in (G, T).

A T-*pairing* is a partition of T into pairs. The value of a T-pairing $\mathcal{P}$ is defined as:

$$val_{G,c}(\mathcal{P}) = \min_{\{u,v\} \in \mathcal{P}} \lambda_{G,\{u,v\},c}$$

Let $\mathcal{P}$ be any T-pairing and $\delta(S)$ be any T-cut. Since $|S \cap T|$ is odd, $\mathcal{P}$ contains a pair $\{u, v\}$ such that $\delta(S)$ is a $\{u, v\}$-cut. Therefore, $c(\delta(S)) \geq \lambda_{G,\{u,v\},c} \geq val_{G,c}(\mathcal{P})$ and the value of $\mathcal{P}$ is a lower bound on $\lambda_{G,T,c}$. Our main result is the following.

Theorem 1.1. *The maximum value of a T-pairing equals the minimum cost of a T-cut.*

Theorem 1.1 is proven in Section 2. In Section 3, we give some negative results for the minimal TDI system of the T-cut polyhedron. These exclude the possibility of characterizing $\lambda_{G,T,c}$ along the lines of the characterization of optimal T-joins in terms of T-borders given by Sebő [12]. We conclude this introduction with an application of Theorem 1.1 and a deep result of Nash-Williams. (Further applications and extensions of Theorem 1.1 can be found in [3].)

Theorem 1.2 (Nash-Williams [7]). *Every undirected graph G has an orientation $\vec{G}$ such that for every ordered pair of nodes (x, y) there are $\lfloor \lambda_{G,\{x,y\}}/2 \rfloor$ arc-disjoint paths in $\vec{G}$ from x to y.*

When $D = (V, A)$ is a digraph and $S \subset V$, then $d_D^+(S)$ denotes the number of arcs in A with tail in S and head in $V \setminus S$.

Corollary 1.3. *For any graft (G, T), there exists an orientation D of G such that $d_D^+(S) \geq \lfloor \lambda_{G,T}/2 \rfloor$ for every subset S of V with $|S \cap T|$ odd.*

Proof: Given a graft (G, T), let $\mathcal{P}$ be a T-pairing of value $\lambda_{G,T}$. By Theorem 1.2, there exists an orientation $D(V, A)$ of $G(V, E)$ with the property that for every pair $P \in \mathcal{P}$ there are $\lfloor \lambda_{G,T}/2 \rfloor$ arc-disjoint paths from whichever node in P to the other. Consider now any subset S of V with $|S \cap T|$ odd: by parity reasons, there must exist some pair $\{s, t\} \in \mathcal{P}$ with $s \in S$ and $t \notin S$. Hence, D contains $\lfloor \lambda_{G,T}/2 \rfloor$ arc-disjoint paths from s to t and $d_D^+(S) \geq \lfloor \lambda_{G,T}/2 \rfloor$ follows. □

2 Gomory-Hu trees and minimum T-cuts

Given a graft (G, T), a *T-join* is an edge subset $J \subseteq E$ such that every node $v \in V$ is incident with an odd number of edges in J iff $v \in T$. Clearly, when J is a T-join and C is a T-cut, then $|J \cap C|$ is odd. As an example, when G is a tree, then (G, T) admits a unique T-join J, and J contains precisely those edges e of G such that $\{e\}$ is a T-cut.

Given a pair (G, c), let H be a tree with $V(H) = V$ and w_h be a non-negative weight assigned to every edge h of H. The pair (H, w) is a *Gomory-Hu tree* of (G, c) if for every two nodes $u, v \in V$ the following property holds: if h is any edge of minimum weight in the unique path between u and v in H, and where S_h denotes any of the two connected components in the graph obtained from H by deleting h, then $\delta_G(S_h)$ is a minimum $\{u, v\}$-cut for (G, c) and $c(\delta_G(S_h)) = w_h$. In [5], Gomory and Hu proved that a Gomory-Hu tree always exists and showed how to construct one efficiently. In [8], Padberg and Rao gave an algorithm to find minimum T-cuts. Their algorithm was based on the following observation.

Lemma 2.1. *Let (H, w) be a Gomory-Hu tree for (G, c). Let J be the T-join of H. Let h be an edge of J such that $w_h = \min_{e \in J} w_e$. Then $\delta_G(S_h)$ is a minimum T-cut.*

Proof: Let $\delta_G(S)$ be a minimum T-cut. Let $\bar{h} = uv$ be an edge in the T-cut $\delta_J(S)$ of J. We claim that $\delta_G(S_{\bar{h}})$ is a minimum T-cut. Indeed, $|S_{\bar{h}} \cap T|$ is odd since $\bar{h} \in J$. Moreover, $\delta_G(S_{\bar{h}})$ is a minimum $\{u, v\}$-cut since $\bar{h}$ is an edge of H. Hence, $c(\delta_G(S_{\bar{h}})) \leq c(\delta_G(S))$. □

Proof of Theorem 1.1: Let (H, w) be a Gomory-Hu tree for (G, c). Let J be the T-join of H. By Lemma 2.1, $\lambda_{G,T,c} = \min_{h \in J} w_h$. Clearly, every component of (V, J) has an even number of nodes in T. Construct a T-pairing $\mathcal{P}$ by arbitrarily pairing up the nodes of T inside the components of (V, J). Clearly, $val_{G,c}(\mathcal{P}) = \lambda_{G,T,c}$ and Theorem 1.1 follows. □

The clutter of minimal T-joins and the clutter of minimal T-cuts are easily checked to be the blocker of each other. In [2], Edmonds and Johnson showed this pair [6, 4] of clutters to be ideal.

3 On the minimal TDI system for the T-cut polyhedron

Given a graft (G,T), let $P_{G,T}$ denote the T-cut polyhedron (the dominant of the T-cut polytope). Note that $P_{G,T}$ is of blocking type and hence has full dimension. It follows [11] that $P_{G,T}$ admits a unique (up to scaling) minimal TDI system $\mathcal{S}_{G,T}$. In this section, we show that the coefficients in $\mathcal{S}_{G,T}$ can be arbitrarily big. This contrasts with the nice results on the minimal TDI system for the T-join polyhedron given in [12]. More precisely, a *T-border* of a graft (G,T) is a subset B of E such that all components of $G \setminus B$ have an odd number of nodes in T. Denote by $o(B)$ the number of components in $G \setminus B$. It is easy to see that, when $\mathcal{B}$ is a *packing of T-borders* (i.e. a collection of disjoint T-borders), then $\sum_{B\in\mathcal{B}} \frac{o(B)}{2}$ gives a lower bound on the size of a minimum T-join. It is shown in [12] that for every graft (G,T) and for every cost function $c : E \mapsto \mathbb{N}$ there always exists a T-join J and a collection $\mathcal{B}$ of T-borders (repetition is allowed), with no edge e in more than c_e T-borders of $\mathcal{B}$, and such that $c(J) = \sum_{B\in\mathcal{B}} \frac{o(B)}{2}$. Clearly, this nice min-max characterization is only possible because the left side coefficients of the minimal TDI system for the T-join polyhedron are all 0-1.

Let $P \subseteq \mathbb{R}^E$ be a non-empty polyhedron of blocking type. A non-negative integral function $w : E \mapsto \mathbb{N}$ is called a *weighting on E* and the *rank* of w is defined as $r(w) = \min\{wx : x \in P\}$. Clearly, the following infinite system defines P.

$$\left\{\begin{array}{rl} wx \geq & r(w) \quad \forall w : E \mapsto \mathbb{N} \\ x \geq & \mathbf{0} \end{array}\right. \tag{1}$$

Moreover, System 1 is integral and TDI. The minimal TDI system $\mathcal{S}_P$ of P is therefore a subsystem of System 1. A weighting w is *closed* if $r(w') < r(w)$ for every weighting $w' : E \mapsto \mathbb{N}$ with $w' \leq w$ and $w' \neq w$. A separation of w is a pair $w_1, w_2 : E \mapsto \mathbb{N}$ such that $w_1 + w_2 = w$ and $r(w_1) + r(w_2) = r(w)$. If w does not admit any separation with $w_1, w_2 \neq \mathbf{0}$ then w is said *non-separable*. As shown in [9], the proof of Lemma 2.1 in [1] can be easily adapted to obtain the following characterization of those constraints which are in $\mathcal{S}_P$.

Lemma 3.1. *An inequality of the form $wx \geq r(w)$ is in $\mathcal{S}_P$ if and only if $w \neq \mathbf{0}$ is closed and non-separable.*

Proof: If $wx \geq r(w)$ is in the minimal TDI system for P then w is clearly closed and non-separable. To see the converse, let $\bar{w} : E \mapsto \mathbb{N}$ be a closed non-separable weighting. Consider the linear programs:

PRIMAL:

$$\min \bar{w}x \quad \left\{\begin{array}{rl} wx \geq & r(w) \quad \forall w : E \mapsto \mathbb{N} \\ x \geq & \mathbf{0} \end{array}\right.$$

DUAL:

$$\max \sum_{w:E\mapsto\mathbb{N}} r(w)y_w \quad \left\{\begin{array}{rl} \sum_{w:E\mapsto\mathbb{N}} wy_w \leq & \bar{w} \\ y \geq & \mathbf{0} \end{array}\right.$$

The dual has the following integral solution:

$$y_{\bar{w}} = 1, \qquad y_w = 0 \quad \forall w \neq \bar{w}$$

to which corresponds the optimum value $r(\bar{w})$. It suffices to show that no integral optimal dual solution has $y_{\bar{w}} = 0$. Suppose such a solution does exist. To it correspond $w_1, w_2, \ldots, w_j$ not necessarily distinct such that $\bar{w} \geq w_1 + w_2 + \ldots + w_j$ and $r(w_1) + r(w_2) + \ldots + r(w_j) = r(\bar{w})$.

Since $\bar{w}$ is closed and $y_{\bar{w}} = 0$ then $j \geq 2$. But then $\bar{w}$ is separable. $\square$

A graph G is called an r-graph if there exist rk 1-factors of G (repetition allowed) such that every edge of G is contained in precisely k of them. An r-graph is called *indecomposable* when its edge set can not be partitioned as $E_1 \cup E_2$ so that $G_i(V, E_i)$ is an r_i-graph for $i = 1, 2$ and for some $r_1, r_2 > 0$. In [9, 10], indecomposable r-graphs were shown to exist for every r.

Claim 3.2. *For every integer r, there exists a graft (G, T) such that $\mathcal{S}_{G,T}$ contains an inequality $ax \leq b$ with $b \geq r$.*

Proof: Let G be an indecomposable r-graph. We show that the graft (G, V) satisfies the claim. Let $\mathbf{1}$ be the all 1's weighting on E. Then $r(\mathbf{1}) = r$ since G is a r-graph. Moreover, $\mathbf{1}$ is closed since every edge of G is contained in a (trivial) minimum V-cut of $(G, V, \mathbf{1})$. Finally, $\mathbf{1}$ is non-separable since G is indecomposable. Apply Lemma 3.1. $\square$

Claim 3.3. *For every integer r, there exists a graft (G, T) such that $\mathcal{S}_{G,T}$ contains an inequality $ax \leq b$ with a left side coefficient equal to r.*

Proof: Let G be an indecomposable r-graph. Let H be the graph obtained from G by adding two new nodes u and v and an edge uv. We show that the graft $(H, V(H))$ satisfies the claim. Let $\bar{w}$ be the weighting on $E(H)$ defined by $\bar{w}(uv) = r$ and $\bar{w}(e) = 1$ for every $e \in E(G)$. Then, $\bar{w}$ is closed and non-separable and $r = r(\bar{w})$. Apply Lemma 3.1. $\square$

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