A short proof of König's matching theorem

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December 6, 1999

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Abstract

We give a short proof of the following basic fact in matching theory: in a bipartite graph the maximum size of a matching equals the minimum size of a node cover.

Key words: bipartite graphs, maximum matching, minimum node cover.

A *matching* of a graph $G(V,E)$ is a subset M of E such that every node of G is incident with at most one edge in M. A *cover* of G is a set of nodes W such that $G \setminus W$ has no edges. Denote by $\nu(G)$ the maximum cardinality of a matching of G and by $\tau(G)$ the minimum cardinality of a cover of G. Clearly, $\nu(G) \leq \tau(G)$.

We give a short proof of the following basic fact [1] in matching theory.

Theorem. *Let G be a bipartite graph. Then $\nu(G) = \tau(G)$*

Proof: Let G be a minimal counterexample. Then G is connected, is not a circuit, nor a path. So, G has a node of degree at least 3. Let u be such a node and v one of its neighbors. If $\nu(G \setminus v) < \nu(G)$, then, by minimality, $G \setminus v$ has a cover W' with $|W'| < \nu(G)$. Hence, $W' \cup \{v\}$ is a cover of G with cardinality $\nu(G)$ at most. Assume, therefore, there exists a maximum matching M of G having no edge incident at v. Let f be an edge of $G \setminus M$ incident at u but not at v. Let W' be a cover of $G \setminus f$ with $|W'| = \nu(G)$. Since no edge of M is incident at v, it follows that W' does not contain v. So W' contains u and is a cover of G. $\square$

The same proof easily extends to Egerváry's generalization [2] of König's result to graphs with nonnegative weights on the edges.

References

[1] D. König, Graphs and matrices, Mat Fiz Lapok 38 (1931), 116–119 (in Hungarian).

[2] E. Egerváry, On combinatorial properties of matrices, Mat Lapok 38 (1931), 16–28 (in Hungarian).

*Research carried out with financial support of the project TMR-DONET nr. ERB FMRX-CT98-0202 of the European Community