On 4-connected graphs without even cycle decompositions

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Abstract

A *circuit decomposition* of a graph $G = (V, E)$ is a partition of E into circuits. A decomposition is said *even* if all its circuits have even length.

We give a negative answer to a question posed by Jackson asking whether K_5 is the only 4-connected eulerian graph with an even number of edges but no even circuit decomposition.

Key words: even circuit decomposition, eulerian graph, K_5.

1 Introduction

In a graph $G = (V, E)$, a *circuit* is a closed simple path. We regard a circuit C as the set of its edges and say that C is *even* (*odd*) when $|C|$ is even (odd). A *circuit decomposition* of G is a partition of E into circuits. A decomposition is said *even* if all its circuits are even. Clearly, a graph admits a circuit decomposition if and only if it is *eulerian*, i.e. every node has even degree. Therefore, for G to admit an even circuit decomposition, G must be eulerian and $|E|$ must be even. However, K_5, the complete graph on 5 nodes, is an eulerian graph with an even number of edges but no even circuit decomposition. On the other hand, the following result of Zhang [8] extends a previous result of Seymour [7] on planar graphs.

Theorem 1.1 (Zhang – 1994) *Every 2-connected eulerian graph with an even number of edges and no subgraph contractible to K_5 admits an even circuit decomposition.*

In [3], Jackson gave an infinite family of 3-connected eulerian graphs with an even number of edges but no even circuit decomposition, hence contradicting a conjecture of Zhang [9], stating that K_5 was the only such graph. In the same paper, Jackson conjectured however that rising the connectivity to 4 would have made the claim true. This question also appeared as Problem 11.6.10 in [10].

Problem 1.2 (Jackson, Zhang) *Is K_5 the only 4-connected eulerian graph with an even number of edges but no even circuit decomposition?*

In this paper, we answer the above question in the negative by explicitly constructing a 4-connected eulerian graph $\mathcal{G}$ other than K_5, with an even number of edges but no even circuit decomposition. From our arguments, it will also be evident that infinitely many of

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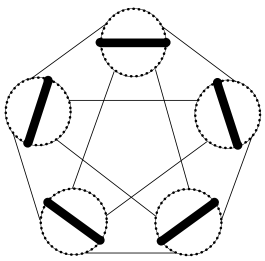
such graphs do actually exist.

In obtaining $\mathcal{G}$, we will follow the same general approach as in [6, 4, 5]. In this spirit, a *gadget* is a configuration with demanding requirements on an hypothetical even circuit decomposition. The second ingredient in the approach is the *skeleton*, which acts like a map telling how the distinct gadgets are mutually connected. In our case, the skeleton $\mathcal{S}$ is a K_5 with a pairing of the edges incident at every node, as appears in Fig. 1. It is well know that $\mathcal{S}$ enjoys the following property:

Fact 1.3 *The following object does not exist in $\mathcal{S}$: a circuit decomposition of the K_5 such that every circuit in the decomposition takes at most one edge per pair at every node.*

Indeed, such a circuit decomposition should be even since the edges in its circuits would alternate between the outer pentagon and the inner star.



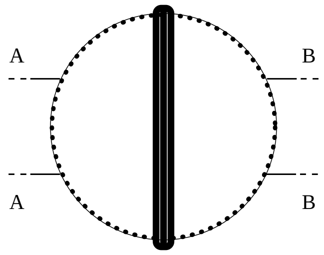


Figure 1: The skeleton, i.e. the top-level diagram of the counterexample.

The counterexample is obtained by exhibiting a gadget, i.e. a subgraph with two pairs of outgoing edges and forcing, on any hypothetical even circuit decomposition, the "at most one edge per pair" condition assumed in Fact 1.3.

2 The gadget

Our gadget is shown in Fig. 2, together with its symbolic representation. Four *pendants* exit the gadget. The pendants [2], also called *semiedges* [1, 5], serve as place-marks of the connections among gadgets, when the gadgets substitute their symbolic representation in the skeleton structure. In our case, the four pendants come into pairs. Letters A and B, but also the vertical bar in the symbolic representation of the gadget, serve to indicate the pairing.

The next lemma establishes the local conditions imposed by a gadget configuration on an hypothetical even circuit decomposition.

Lemma 2.1 *Consider a gadget configuration in a graph $\mathcal{G}$. Let $\{a_1, a_2\}$ and $\{b_1, b_2\}$ be the pairs of edges of $\mathcal{G}$ corresponding to the pairs of pendants of the gadget. Let $\mathcal{C}$ be an even circuit decomposition of $\mathcal{G}$. Then, in $\mathcal{C}$ there are precisely two circuits (say C_1 and C_2) containing edges in $\{a_1, a_2, b_1, b_2\}$. Moreover, both C_1 and C_2 have one edge in $\{a_1, a_2\}$ and one in $\{b_1, b_2\}$.*

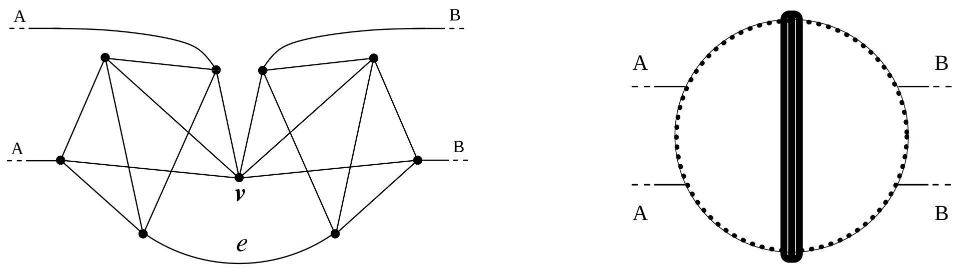


Figure 2: The gadget and its symbolic representation.

Note that each gadget has an odd number of edges, and hence, had $\mathcal{G}$ to be obtained precisely as indicated in Fig. 1, then $|E(\mathcal{G})|$ would not be even and $\mathcal{G}$ would not be a counterexample. It is not difficult however to refine the gadget as to have an even number of edges, while preserving the properties expressed by Lemma 2.1. This can be achieved for example by just chaining two gadgets as in Fig. 3. Clearly, if an odd total number of chainings occur, then $|E(\mathcal{G})|$ is even. Moreover, it is easy to verify that $\mathcal{G}$ is indeed 4-connected and eulerian. Finally, by Fact 1.3 and Lemma 2.1, $\mathcal{G}$ has no even circuit decomposition and hence is a counterexample.

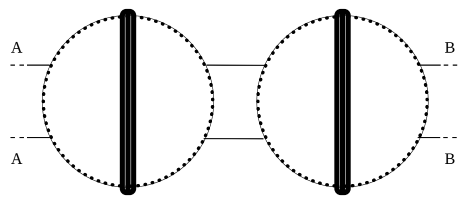


Figure 3: Chaining two gadgets yields a gadget with an even number of edges.

Verification of Lemma 2.1, while being in principle only a finite problem, will take the remaining part of this section. The specular symmetry on the vertical axis suggests to regard the gadget displayed in Fig. 2 as a pair of *twins* joined on the central node v. As suggested in Fig. 4, a twin is a close relative of K_5. To prove Lemma 2.1, we first consider the local conditions imposed by a single twin on an hypothetical even circuit decomposition. Note that, in the case of the twin, some non-twin edges can have an endpoint in v.

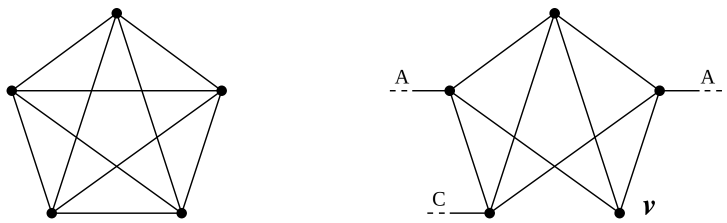


Figure 4: Graph K_5 and a twin of the gadget.

Lemma 2.2 *Consider a twin in a graph $\mathcal{G}$. Let a_1, a_2 be the two edges of $\mathcal{G}$ corresponding to the pair of pendants labelled with A in Fig. 4. Let c be the edge corresponding to the remaining pendant. Let $\mathcal{C}$ be an even circuit decomposition of $\mathcal{G}$. Then either a_1, a_2 and c are in three different circuits of $\mathcal{C}$ or (i), (ii) and (iii) here below hold:*

(i) in $\mathcal{C}$ there are precisely two circuits (say C_1 and C_2) containing edges in $\{a_1, a_2, c\}$;

(ii) C_1 and C_2 have a node in common inside the twin;

(iii) if $\{a_1, a_2\}$ is contained in C_1 or in C_2 then the number of edges of C_1 inside the twin and the number of edges of C_2 inside the twin are both even.

Proof: Assume in $\mathcal{C}$ there exist at most two circuits with an edge in $\{a_1, a_2, c\}$. Then, inside the twin, $\mathcal{C}$ consists of a (possibly empty) family $\mathcal{C}_t$ of even circuits of the twin plus two paths P_v and P of the twin, where P_v has an endnode in v (see Fig. 4). Since the twin has an even number of edges and all circuits in $\mathcal{C}_t$ are even then $|E(P_v)|$ and $|E(P)|$ have the same parity. To show (i) and (ii) we must show that P_v and P have a node in common. This is certainly true if one of the two paths has length at least three since in the twin there exists a single node which is neither an endnode of P_v nor an endnode of P. Indeed, this is also true if one of the two paths has length at least two since the lengths of the two paths have the same parity. So assume $|E(P_v)| = |E(P)| = 1$. But then, removing the edges of P_v and P from the twin, we remain with a graph $\mathcal{H}$ which consists of two triangles with exactly one node in common. Graph $\mathcal{H}$ has no even circuit decomposition (and is not 3-connected). However $\mathcal{C}_t$ is an even circuit decomposition of $\mathcal{H}$ — a contradiction.

It only remains to show (iii). Assume by absurd that edge c has an endnode u in common with P_v and $|E(P_v)|$ is odd. Denote by h and k the endnodes of P. Note now that the graph obtained from the twin by removing the three pendants and adding an edge uv and an edge hk is a K_5. Consider the circuits $C_v = E(P_v) \cup \{uv\}$ and $C = E(P) \cup \{hk\}$. Now, $|E(C_v)|$ and $|E(C)|$ are even and $\mathcal{C}_t \cup \{C_v\} \cup \{C\}$ is an even circuit decomposition of K_5 — a contradiction. $\square$

Proof of Lemma 2.1: Assume on the contrary that $\{a_1, a_2\}$ is contained in a circuit C_A of $\mathcal{C}$. But then $\{b_1, b_2\}$ is contained in a circuit C_B of $\mathcal{C}$, with C_B possibly the same as C_A. However, by Lemma 2.2 (i), the circuit $\bar{C}$ of $\mathcal{C}$ containing the edge e from Fig. 2 is neither the same as C_A nor the same as C_B. By Lemma 2.2 (iii), $|\bar{C}|$ is odd — a contradiction. $\square$

3 A new conjecture

In view of our constructions it seems now natural to consider the following conjecture.

Conjecture 3.1 *Every 2-connected eulerian graph with an even number of edges and other than K_5 has an even circuit decomposition or a cut of size at most 4 which is not the star of a node.*

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